CSM – 52 / 15
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

*Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and **three** of the remaining questions, selecting at least **one** from each Section.*

Section – A

1. Answer any **three** of the following :

   (a) Prove that the order of a subgroup H of a finite group G divides the order of G. 20

   (b) In a vector space V of dimension n, prove that a subset of n vectors is linearly independent if and only if it spans V. 20

(c) Find a matrix of order 3 whose eigenvalues are 1, 2, 3 and the corresponding eigen-vectors are $\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}$ and $\begin{pmatrix}0\\1\\1\end{pmatrix}$ respectively.

20

(d) Show that the equation : 20

$$\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$$

represents a pair of planes.

2. (a) If x and y are relatively prime and not both zero, show that there are integers m and n such that the greatest common divisor of x and y equals mx + ny. 15

(b) If G is a finite group and p is a prime number such that p^{α} divides o(G), show that there is a subgroup of G of order p^{α}. 15

(c) If G is an Abelian group, show that $(a \cdot b)^n = a^n \cdot b^n$ for all integers n and for all a, b, ∈ G. Show that converse is true if $(a \cdot b)^n = a^n \cdot b^n$ for three consecutive integers n and for all a, b ∈ G. 15

(d) Let R be a commutative ring with unit element and M is an ideal of R. Show that M is a maximal ideal if and only if R/M is a field. 15

3. (a) Let V be a vector space of dimension n over a field F. Show that the transformation that maps a vector $v \in V$ to the corresponding coordinate vector with respect to a fixed basis of V is a linear transformation from V to F^n. 15

(b) If V is a finite-dimensional vector space and if W is a subspace of V, show that W is also finite-dimensional and $\dim W \leq \dim V$ and $\dim(V/W) = \dim V - \dim W$. Show that $\dim W = \dim V$ if and only if W = V. 15

(c) Show that the eigenvalues of a Hermitian matrix are real. Deduce that the eigenvalues of a symmetric matrix with real entries are real. 15

(d) Let A be a matrix of order n and let adj A denote the adjoint matrix of A. Show that : 15

(i) A is invertible if and only if $A^T A$ is invertible.

(ii) A is invertible if and only if adj A is invertible.

(iii) The determinant of adj A equals $|A|^{n-1}$.

4. (a) Tangents are drawn from the point P(h, k) to the circle $x^2 + y^2 = a^2$. These tangent touch the circle at A, B. Find the area of the triangle ABP. 15

(b) Find the value of λ so that the equation $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$ may represents a pair of straight lines. Find the lines. 15

(c) Let a point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Prove that its locus is a sphere. 15

(d) Find the condition so that the plane $\ell x + my + nz = p$ may touch the central conicoid $ax^2 + by^2 + cz^2 = 1$. 15

Section – B

5. Answer any **three** of the following :

(a) Show that the interval [0, 1] is uncountable.

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(b) If $f(z) = u + iv$ is analytic and $u^4 + uv = 2016$, show that f is a constant. 20

(c) Evaluate the integral :

$$\oint_C \left(\bar{z} + \frac{1}{z}\right)^3 dz$$

where C is the circle $|z| = 1$ in the anticlockwise direction. 20

(d) Show that the series $\sum_{n=1}^{\infty} n^{-p}$ converges for $p > 1$ and diverges for $0 < p \leq 1$. 20

6. (a) Show that a monotonically increasing function on an interval [0, 1] is Riemann integrable. 15

(b) Suppose that the sequence $<f_n>$ of Riemann integrable functions of [0, 1] converges to a function f. Show that the function f is Riemann integrable and

$$\int_0^1 f(x)dx = \lim_{n\to\infty} \int_0^1 f_n(x)dx. \qquad 15$$

(c) Find the Laurent series expansion of the function $f(z) = 1/(z^2 - 3z + 2)$ valid in : 15

(i) $1 < |z| < 2$

(ii) $|z-1|<1$

(iii) $1<|z-1|<2$

(iv) $|z-2|<1$

(d) Evaluate the integral

$$\oint_C \frac{dz}{z^4-3z^2-4}$$

where C is the circle $|z-1|=2$ in the anticlockwise direction. 15

7. (a) If $\alpha>1$, show that the inequality $(1+x)^{\alpha}\geq 1+\alpha x$ holds for all $x>-1$. Discuss the case of equality. 15

(b) Express the integral $\int_0^1 x^m(\log x)^n dx$ when $m, n>0$ in terms of gamma function. 15

(c) Find the area of the surface of the solid generated by revolving astroid $x=a\cos^3 t$, $y=a\sin^3 t$ about the axis of x. 15

(d) Evaluate the triple integral

$$\iiint_G xy^2z^3dV$$

over the rectangular box G defined by the inequalities $-1 \leq x \leq 2$, $0 \leq y \leq 3$ and $0 \leq z \leq 2$. 15

8. (a) Show that div(curl F) = 0 and curl($\nabla\phi$) = 0. Verify these results directly when $F(x, y, z) = x^2y\hat{i} + 2y^3z\hat{j} + 3z\hat{k}$ and $\phi(x, y, z) = x^2y + y^2z + z^2x$. 15

(b) Find the work done by the force-field $F(x, y, z) = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$ on a particle that travels once around the unit circle $x^2 + y^2 = 1$ in the clockwise direction. 15

(c) Using Gauss divergence theorem, find the outward flux of the vector field $F(x, y, z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ across the surface of the region that is enclosed by the circular cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 2$. 15

(d) Let $F(x, y, z) = 2z\hat{i} + 3x\hat{j} + 5y\hat{k}$, σ be the portion of the paraboloid $z = 4 - x^2 - y^2$ for which $z \geq 0$ with upward orientation and C be the positively oriented circle $x^2 + y^2 = 4$

that forms the boundary of σ in the xy-plane.

Compute $\oint_C F . dr$ and $\iint_\sigma (\text{curl } F) . n dS$. Does this verify Stokes' theorem ? 15