MATHEMATICS

Time allowed : Three Hours

Full Marks : 100

All questions carry equal marks.

*Answer any **FIVE** questions.*

(Symbols used have their usual meaning)

1. (a) If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that :

   (i) $a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots$ and

   (ii) $\lim_{n \to \infty} a_n = 0$ show that the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent. 8

   (b) (i) Define (C, 2) summability. Show that : 1, –1, 2, –2, 3, – 3, … is not (C, 1) summable but (C, 2) summable. 6

   (ii) Determine all real values of x for which the following series converges :

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\frac{\sin nx}{n}.$$ 6

2. (a) Let f(z) be an analytic function in a simply connected domain D enclosed by a rectifiable Jordan curve

C and let f(x) be continuous on C. Show that

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z - z_0}\,dt$$

where z_0 is any point in D. 8

(b) (i) Find the bilinear transformation which maps the points $z_1 = 2$, $z_2 = i$ and $z_3 = -2$ into the points $w_1 = 1$, $w_2 = i$ and $w_3 = -1$. 6

(ii) Represent the function $f(z) = \frac{z}{(z-1)(z-3)}$ by a series of negative powers of $(z - 1)$ which converges to $f(z)$ when $0 < |z - 1| < 2$. 6

3. (a) State and prove Lebesgue's monotone converge theorem. 8

(b) (i) Construct a non-measurable set on the real line. 6

(ii) Prove that the function $\frac{\sin x}{x}$ is not Lebesgue integrable over $[0, \infty)$. 6

4. (a) State and prove Gauss-Bonnet Theorem on the total curvature. 8

(b) (i) On the paraboloid $x^2 - y^2 = z$, find the orthogonal trajectories of the sections by the planes z = constant. 6

(ii) Obtain the curvature and torsion of the curve of intersection of two quadratic surfaces :

$ax^2 + by^2 + cz^2 = 1, \; a'x^2 + b'y^2 + c'z^2 = 1$ 6

5. (a) Using Gaussian elimination find the inverse of the matrix :

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$$ 8

(b) (i) Using a polynomial of the third degree, complete the record given below of the export of a certain commodity during the 5 years. 6

Year	2005	2006	2007	2008	2009
Export (in tons)	443	384	——	397	467

(ii) Using the fourth order Runge-Kutta method, find an approximate value of y when x = 0.2 given that $y' = x + y$, $y(0) = 1$. 6

6. (a) Let X be a complete metric space, and let $\{F_n\}$ be a decreasing sequence of non-empty closed subsets of X such that $d(F_n) \to 0$. Show that $F = \bigcap_{n=1}^{\infty} F_n$ contains exactly one point. 8

(b) (i) Let X be a second countable space. If a non-empty open set G in X is represented as the union of a class $\{G_i\}$ of open sets, show that G can be represented as a countable union of G_i's. 6

(ii) Prove that any continuous image of a connected space is connected. 6

7. (a) State and prove the Closed Graph Theorem. 8

(b) (i) Prove that a normed space X is finite dimensional if and only if the closed unit sphere in X is compact. 6

(ii) Consider the Banach space $(C[0, 1], \|\cdot\|_\infty)$ and the normed space $(C'[0, 1], \|\cdot\|_\infty)$ where $\|\cdot\|_\infty$ is the sub norm. Define the mapping $T : C'[0, 1] \rightarrow C[0, 1]$ by $(Tx)(t) = x'(t)$, $x \in C''[0, 1]$. Prove that (1) T is not bounded (2) T is closed. 6

8. (a) Let J(i) denote the set of all complex numbers of the form $a + ib$ where a, b are integers. Prove that J[i] is an Euclidean ring. 8

(b) (i) Prove that the ideal $A = (a_0)$ is a maximal ideal of the Euclidean ring R if and only if a_0 is a prime element of R. 6

(ii) If L is an algebraic extension of K and if K is an algebraic extension of F, show that L is an algebraic extension of F. 6

9. (a) Let V be a finite dimensional vector space over the field F, and let W be a subspace of V. Prove that $\dim W + \dim W^0 = \dim V$. 8

(b) (i) Let A be any $m \times n$ matrix over the field F. Show that the row rank of A is equal to the column rank of A. 6

(ii) If F is a field, and M is any non-zero ideal in F[x], show that there is a unique monic polynomial d in F[x] such that M is the principal ideal generated by d. 6

10. (a) Using Monge's method, solve the following equation :

$$2x^2r - 5xys + 2y^2t + 2(px + qy) = 0.$$ 8

(b) (i) Establish the orthogonal property of Legendre's polynomials. 6

(ii) Find the inverse Laplace transform of :

$$\frac{2s^2 - 6s + 5}{s^3 - 6s^2 + 11s - 6}.$$ 6