JL – 26/14

Mathematics

Paper – II

Time : 3 hours

Full Marks : 200

The figures in the right-hand margin indicate marks.

*Answer **five** questions choosing **one** from each Unit.*

UNIT – I

1. (a) (i) Let $f_n(x) = \dfrac{n + \cos x}{2n + \sin^2 x}$ for all real x. Show that (f_n) converges uniformly on R. 5

(ii) Evaluate $\int_{-2}^{3} [|x|]\, d|x|$ where [x] is the largest integer $\leq x$. 5

(b) What is a measurable set ? Prove that the collection of all measurable sets is a σ-algebra. 15

(c) State and prove Minkowski's inequality for $L^p [0, 1]$ space. 15

2. (a) (i) Suppose f is a real, continuously differentiable function on [a, b], $f(a) = f(b) = 0$ and $\int_a^b f^2(x)\,dx = 1$. Prove that $\int_a^b x\, f(x)\, f'(x)\, dx = -\frac{1}{2}$. 5

(ii) If $f \in L^1[0, 1]$ and $g \in L^\infty[0, 1]$, prove that $\int_0^1 |fg| \le \int_0^1 |f| \cdot \|g\|_\infty$. 5

(b) State Fatou's Lemma. Use Fatou's Lemma to prove Lebesgue's Dominated convergence theorem. 15

(c) Construct a set which is not measurable in the sense of Lebesgue. 15

UNIT – II

3. (a) (i) Let X, Y, Z be metric spaces. If $F : X \to Y$ is continuous and $G : Y \to Z$ is closed, prove that $GOF : X \to Z$ is closed. 5

(ii) Let Y be a subspace of a normed space X. Show that $Y^0 \neq \phi$ if and only if $Y = X$. 5

(b) Let Y be a closed subspace of a normed space X. For x + Y in the quotient space X/Y, let $|||\, x + Y \,||| = \inf\{\| x + y \| : y \in Y\}$.

Show that $|||.|||$ is a norm on X/Y. Prove that a sequence $(x_n + Y)$ converges to $x + Y$ in X/Y if and only if there is a sequence (y_n) in Y such that $(x_n + y_n)$ converges to x in X. 15

(c) Let $<,>$ be an inner product on a linear space X. If $\{u_1, u_2, \cdots\}$ is a countable orthonormal set in X and $x \in X$, then prove that $\sum_n |<x, u_n>|^2 \leq \|x\|^2$ and equality holds if and only if $x = \sum <x, u_n> u_n$. 15

4. (a) (i) Let (x_n) be a weakly convergent sequence in a normed space X. Prove that weak limit of (x_n) is unique. 5

(ii) If $A \neq \phi$ is a subset of an inner product space X, prove that $A^{\perp} = A^{\perp\perp\perp}$. 5

(b) Prove that a normed space X is a Banach space if and only if every absolutely convergent series of elements in X is convergent. 15

(c) Let X be a normed space. If the linear space X' of all continuous linear functionals on X is separable, prove that X is separable. 15

UNIT – III

5. (a) (i) Find the coordinates of the vector (1, 2, 1) relative to the ordered basis $B = \{(2, 1, 0), (2\ 1\ 1), (2, 2, 1)\}$. 5

(ii) Evaluate the rank of the matrix : 5

$$\begin{bmatrix} 37 & 259 & 481 & 407 \\ 19 & 133 & 247 & 209 \\ 25 & 175 & 325 & 275 \end{bmatrix}$$

(b) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Prove that T is diagonalizable by exhibiting a basis for R^3, each vector of which is an eigen-vector of T. 15

(c) If W_1 and W_2 are finite dimensional subspaces of a vector space V, prove that $W_1 + W_2$ is finite dimensional and $\dim W_1 + \dim W_2 = \dim (W_1 \cap W_2) + \dim (W_1 + W_2)$. 15

6. (a) (i) Find the characteristic polynomial and minimal polynomial of the matrix : 5

$$\begin{bmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{bmatrix}$$

(ii) Let T be the linear operator on R^3 defined by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$. Is T invertible ? If so, find a rule for T^{-1} like the one which defines T. 5

(b) Let V and W be vector spaces over the field F and let T be a linear transformation from V to W. If V is finite dimensional, prove that rank(T) + nullity(T) = dim(V). 15

(c) Let V be a vector space over the field F and let T be a linear operator on V. If $\lambda_1, \lambda_2, \ldots\ldots \lambda_k$ in F are distinct eigenvalues of T and if $\upsilon_1, \upsilon_2 \ldots\ldots\ldots \upsilon_k$ are eigenvectors of T belonging to $\lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_k$ respectively, prove that $\upsilon_1, \upsilon_2 \ldots\ldots\ldots \upsilon_k$ are linearly independent over F. 15

UNIT – IV

7. (a) (i) Prove by mathematical induction that $(11)^{n+2} + (12)^{2n+1}$ is divisible by 133 for every positive integer n. 5

(ii) A graph G has 14 vertices and 27 edges. The degree of every vertex of G is 3, 4 or 5. There are six vertices of degree 4. How many vertices of G have degree 3 and how many have degree 5 ? 5

(b) (i) Show that $R \to S$ can be derived from the premises $P \to (Q \to S)$, $\urcorner R \vee P$ and Q. 7

(ii) Obtain the principal disjunctive normal form of $P \to (P \wedge (Q \to P))$. 8

(c) Find an explicit formula for the Fibonacci sequence. 15

8. (a) (i) Use a K-map to find a minimal expansion as a Boolean sum of Boolean product of the function $F(x, y, z) = xyz + xy\overline{z} + x\overline{y}z + x\overline{y}\,\overline{z} + \overline{x}yz + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}$, where $\overline{x}$ denotes the complement of x. 5

(ii) Let X = {a, b, c} and P(X) its power set. Let $\subseteq$ be the inclusion relation on the elements of P(X). Draw the Hasse diagram of the partially ordered set $\langle P(X), \subseteq \rangle$. 5

(b) What is a tree ? Prove that a tree with n vertices has precisely n – 1 edges. Draw all non-isomorphic trees with 5 vertices. 15

(c) (i) Show that in a complemented, distributive lattice $\langle L, *, \oplus \rangle$

$$a \le b \Leftrightarrow a * b' = 0 \Leftrightarrow a' \oplus b = 1 \Leftrightarrow b' \le a'$$

for $a, b \in L$. 8

(ii) Show that every distributive lattice is modular. 7

UNIT – V

9. (a) (i) Solve the initial value problem : 5

$$\frac{dy}{dx} + 5y = 3e^x, \; y(0) = 1.$$

(ii) Solve $\frac{y^2 z}{x} \frac{\delta z}{\delta x} + 2x \frac{\delta z}{\delta y} = y^2.$ 5

(b) Find the complete integral of the equation

$$\left(\left(\frac{\delta z}{\delta x}\right)^2 + \left(\frac{\delta z}{\delta y}\right)^2\right) y = z \frac{\delta z}{\delta y}$$ by Charpit's method. 15

(c) Prove the recurrence relation $\frac{d}{dx} P_n(x) - x \frac{d}{dx} P_{n-1}(x) = nP_{n-1}(x)$, where $P_n(x)$ is the Legendre polynomial of degree n. 15

10. (a) (i) Find the Fourier series on the interval $-\pi < x < \pi$ that corresponds to the function f defined by the equation : 5

$$f(x) = \begin{cases} -\frac{\pi}{2} & \text{when } -\pi < x < 0 \\ \frac{\pi}{2} & \text{when } 0 \le x < \pi \end{cases}$$

(ii) Find the particular integral of the equation $(D^2 + 2DD' + D'^2)z = 12xy$, where $D = \frac{\delta}{\delta x}$ and $D' = \frac{\delta}{\delta y}$. 5

(b) Use Laplace transform to solve the initial value problem : 15

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 3y = 0,\ y(0) = 1,\ y'(0) = 7$$

(c) Solve $\frac{\delta^2 z}{\delta x^2} - \frac{\delta^2 z}{\delta y^2} = 0$ by Monge's method. 15